Modal Information Logics: Axiomatizations and Decidability

Søren Brinck Knudstorp
Based on MSc thesis, supervised by Johan van Benthem and Nick Bezhanishvili
October 18, 2023

ILLC, University of Amsterdam

Plan for the talk

- Big picture, few details (so please let me know if you'd like elaboration)
- Outline of the talk
  1. Introducing the logics
  2. Stating the problems
  3. Outlining the strategy
  4. Solving the problems using the strategy
- Overarching theme: a study of modal information logics

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Defining (the basic) modal information logics (MILs)

Definition (language and semantics)

The language is given by

$$\varphi ::= \bot \mid p \mid \neg\varphi \mid \varphi \vee \psi \mid \langle \sup \rangle \varphi\psi,$$

and the semantics of '$\langle \sup \rangle$' is:

$$w \Vdash \langle \sup \rangle \varphi\psi \quad \text{iff} \quad \exists u, v (u \Vdash \varphi;\ v \Vdash \psi;\ w = \sup\{u, v\})$$

Example

Definition (frames and logics)

Three classes of frames $(W, \leq)$, namely those where

(Pre) $(W, \leq)$ is a preorder (refl., tr.);

(Pos) $(W, \leq)$ is a poset (anti-sym. preorder); and

(Sem) $(W, \leq)$ is a join-semilattice (poset w. all bin. joins)

Resulting in the logics MIL_{Pre}, MIL_{Pos}, MIL_{Sem}, respectively.

Appetizer: Let's show that $MIL_{Pre} \subseteq MIL_{Pos} \subsetneq MIL_{Sem}$. **see blackboard**

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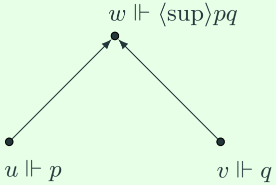

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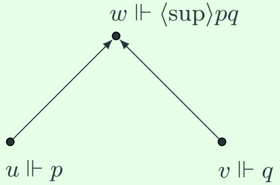

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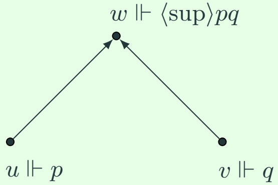

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How can we think of this algebraically?

From relations to algebras

Given a preorder $(W, \leq)$, we can form its complex algebra w.r.t. the induced supremum relation:

$$(\mathcal{P}(W), \cap, \cup, ^c, \varnothing, W, \cdot),$$

where

$$Y \cdot Z := \{x \in W \mid x = \sup\{y, z\}, y \in Y, z \in Z\}.$$

Let Pre^+, Pos^+ and Sem^+ denote the classes of complex algebras of preorders, posets and (join-)semilattices w.r.t. the supremum relation. Then,

- $\mathit{MIL}_{\mathit{Pre}}$ corresponds to the variety $\mathbf{V}(\mathit{Pre}^+)$;
- $\mathit{MIL}_{\mathit{Pos}}$ to the variety $\mathbf{V}(\mathit{Pos}^+)$; and
- $\mathit{MIL}_{\mathit{Sem}}$ to the variety $\mathbf{V}(\mathit{Sem}^+)$.

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Why MILs?

- Connect with other logics (e.g., truthmaker semantics).
- Introduced to model a theory of information (by van Benthem (1996)).
- Modestly extend $\mathbf{S4}$ [MIL_{Pre}, MIL_{Pos}]. **see blackboard**

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Guided by two central problems (posed in van Benthem (2017, 2019)), namely

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(D) proving (un)decidability.

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Proposition

MILs lack the finite model property (FMP) w.r.t. their classes of definition. **see blackboard**

How we solve (A), and then (D) using (A):

(1) We axiomatize MIL_{Pre} (and deduce $MIL_{Pre} = MIL_{Pos}$).

(2) Use the axiomatization to find another class of structures $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = MIL_{Pre}$.

(3) Prove that on $\mathcal{C}$ we do have the FMP and deduce decidability.

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Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

(4) $PPp \rightarrow Pp$

(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

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Proof idea

Soundness *see blackboard* ✓

For completeness, let $\Gamma \supseteq \Gamma_0$ be an MCS extending some consistent Γ_0. We construct a satisfying model using the step-by-step method (but first, why step-by-step? *see blackboard*).

(Base) Singleton frame $\mathbb{F}_0 := (\{x_0\}, \{(x_0, x_0)\})$ and 'labeling' $l_0(x_0) = \Gamma$.

(Ind) Suppose $(\mathbb{F}_n, l_n)$ has been constructed.
– If $x \in \mathbb{F}_n$ and $\neg\langle \text{sup} \rangle \psi\psi' \in l_n(x)$ but $x = \text{sup}_n\{y, z\}$ s.t. $\psi \in l_n(y), \psi' \in l_n(z)$, *coherently* extend to $(\mathbb{F}_{n+1}, l_{n+1}) \supseteq (\mathbb{F}_n, l_n)$ so that $x \neq \text{sup}_{n+1}\{y, z\}$.
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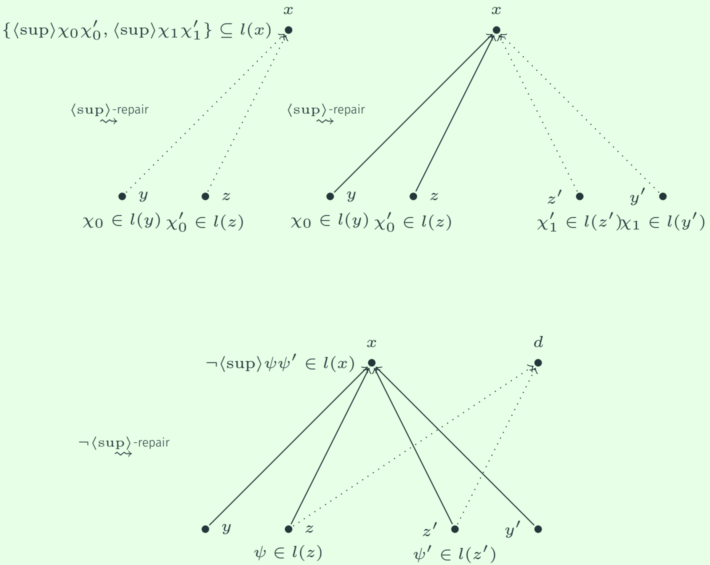
– Similarly, for $\langle\text{sup}\rangle\chi\chi' \in l_n(x)$.

Example


x
$\{\langle \text{sup} \rangle \chi_0 \chi_0', \langle \text{sup} \rangle \chi_1 \chi_1'\} \subseteq l(x)$
$\langle \text{sup} \rangle$-repair
y
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$\chi_0 \in l(y)$ $\chi_0' \in l(z)$
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x
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z'
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(1): axiomatizing MIL_{Pre}

Axiomatization (soundness and completeness)

MIL_{Pre} is (sound and complete w.r.t.) the least normal modal logic with axioms:

(Re.) $p \wedge q \rightarrow \langle \text{sup} \rangle pq$

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(Co.) $\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp$

(Dk.) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq$

About the proof

Soundness: routine.

Completeness: step-by-step method.

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As a corollary we get that $MIL_{Pre} = MIL_{Pos}$.

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(2) Find another class $\mathcal{C}$ for which $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$:

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$$\mathcal{C} := \{(W, C) \mid (W, C) \Vdash (Re.) \land (Co.) \land (4) \land (Dk.)\},$$

  where $C \subseteq W^3$ is an arbitrary relation.

  (iii) Then $\mathrm{Log}(\mathcal{C}) = \mathit{MIL}_{\mathit{Pre}}$. **see blackboard**

(3) Decidability through FMP on $\mathcal{C}$:

  (i) On $\mathcal{C}$, we get the FMP through filtration.

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Thus, we have solved both (A) and (D).

Gen. takeaway: *When dealing with ‘semantically introduced’ logics, not having the FMP (w.r.t. the class of definition) might not be very telling.*

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Can we generalize these techniques?

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(Natural) extensions of MIL_{Pre} and MIL_{Pos} [and $\mathbf{S4}$] are obtained by adding an informational implication '\'.

Definition

The language is given by adding '\' with semantics:

$$v \Vdash \varphi \backslash \psi \qquad \text{iff} \qquad \forall u, w([u \Vdash \varphi, w = \sup\{u, v\}] \Rightarrow w \Vdash \psi)$$

We denote the resulting logics as $MIL_{\backslash\text{-}Pre}, MIL_{\backslash\text{-}Pos}$, respectively.

Note that '$\langle\sup\rangle$' and '\' are "inverses"; and 'F' is expressible: we extend temporal **S4**. **see blackboard**

The problems now become

(A\) axiomatizing $MIL_{\backslash\text{-}Pre}$ and $MIL_{\backslash\text{-}Pos}$; and
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The same (1)-(2)-(3) structure is used as before, but now we

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This concludes and summarizes our study of MILs on preorders and posets.

How about join-semilattices (i.e., MIL_{Sem})?

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Three ways to completeness (some intuitions for our proof):

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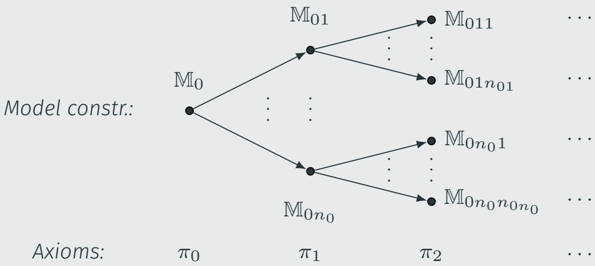

Conclusion and future work

What we have done:

- Surveyed the landscape of MILs on preorders and posets.[1]
- Made crossings with the Lambek Calculus and truthmaker semantics.[2]
- Axiomatized MIL_{Sem}.

What (might) come next while in Chapman:

- Proving (un)decidability of MIL_{Sem} and solving the ancillary problem of fin. axiomatizability
- Proving (un)decidability of Urquhart's relevance logic $\mathbf{S}$
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- Proving (un)decidability of MIL_{Sem} and solving the ancillary problem of fin. axiomatizability
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- *And, who knows, perhaps also working on some ideas of yours :-)*

[1] See SBK (Forthcoming[b])

[2] See SBK (Forthcoming[a]) (or my thesis) for this, including proofs of decidability (and compactness) of a family of truthmaker logics.

Conclusion and future work

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Thank you!

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